

ON THE FIRST EIGENVALUE FOR A $(p(x), q(x))$ -LAPLACIAN ELLIPTIC SYSTEM

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ABSTRACT. In this article, we deal about the first eigenvalue for a nonlinear gradient type elliptic system involving variable exponents growth conditions. Positivity, boundedness and regularity of associated eigenfunctions for auxiliaries systems are established.

1. INTRODUCTION AND SETTING OF THE PROBLEM

In the present paper, we focus on finding a non zero first eigenvalue for the system of quasilinear elliptic equations

$$(1.1) \quad (P) \quad \begin{cases} -\Delta_{p(x)} u = \lambda c(x)(\alpha(x) + 1)|u|^{\alpha(x)-1}u|v|^{\beta(x)+1} & \text{in } \Omega \\ -\Delta_{q(x)} v = \lambda c(x)(\beta(x) + 1)|u|^{\alpha(x)+1}v|v|^{\beta(x)-1} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$. Here $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ and $\Delta_{q(x)} v = \operatorname{div}(|\nabla v|^{q(x)-2} \nabla v)$ are usually named the $p(x)$ -Laplacian and the $q(x)$ -Laplacian operator.

During the last decade, the interest for partial differential equations involving the $p(x)$ -Laplacian operator is increasing. When the exponent variable function $p(\cdot)$ is reduced to be a constant, $\Delta_{p(x)} u$ becomes the well-known p -Laplacian operator $\Delta_p u$. The $p(x)$ -Laplacian operator possesses more complicated nonlinearity than the p -Laplacian. So, one cannot always transpose to the problems arising the $p(x)$ -Laplacian operator the results obtained with the p -Laplacian. The treatments of solving these problem are often very complicated and needs a mathematical tools (Lebesgue and Sobolev spaces with variable exponents, see for instance [4] and its abundant reference). Among them, finding first eigenvalue of $p(x)$ -Laplacian Dirichlet presents more singular phenomena which do not appear in the constant case. More precisely, it is well known that the first eigenvalue for the $p(x)$ -Laplacian Dirichlet problem may be equal to zero (for details, the reader interested can consult [9]). In [9], the authors consider that Ω is a bounded domain and p is a continuous function from $\overline{\Omega}$ to $]1, +\infty[$. They given some geometrical conditions insuring that the first eigenvalue is 0. Otherwise, in one dimensional space, monotonicity assumptions on the function p is a necessary and sufficient condition such that the first eigenvalue is strictly positive. In higher dimensional case, assuming monotonicity of an associated function defined by p , the first eigenvalue is strictly positive.

The fact of the first eigenvalue is zero, has been observed earliest by [7]. Indeed, the authors illustrate this phenomena by taking $\Omega = (-2, 2)$ and $p(x) = 3\chi_{[0,1]}(x) +$

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$(4 - |x|)\chi_{[1,2]}(x)$. In this condition, the Rayleigh quotient

$$\mu_1 = \inf_{u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^{p(x)}}{\int_{\Omega} |u|^{p(x)}}$$

is equal to zero. The main reason derives that the well-known Poincaré inequality is not always fulfilled. However, Fu in [11] shown that when Ω is a bounded Lipschitz domain, p is $L^\infty(\Omega)$ the Poincaré inequality holds (i.e. there is a constant C depending on Ω such that for any $u \in W_0^{1,p(x)}(\Omega)$, $\int_{\Omega} |u|^{p(x)} \leq C \int_{\Omega} |\nabla u|^{p(x)}$). For a use of this result see for instance [2], [12].

Further works established suitable conditions drawing to a non zero first eigenvalue (see [10], [17], [16]).

Compared the investigation for one equation, elliptic systems haven't a similar growth concerning in the first eigenvalue. First of all, when $p(x)$ and $q(x)$ are constant on Ω , in [3], the following elliptic Dirichlet system is considered

$$(1.2) \quad \begin{cases} -\Delta_p u = \lambda |u|^{\alpha-1} |v|^{\beta+1} & \text{in } \Omega \\ -\Delta_q v = \lambda |u|^{\alpha+1} |v|^{\beta+1} v & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Assuming, Ω is a bounded open in \mathbb{R}^N with smooth boundary $\partial\Omega$ and the constant exponents $-1 < \alpha, \beta$ and $1 < p, q < N$ satisfying the condition

$$(1.3) \quad C_{p,q}^{\alpha,\beta} : \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1 \text{ and } (\alpha+1) \frac{N-p}{Np} + (\beta+1) \frac{N-q}{Nq} < 1,$$

the author shown the existence of the first eigenvalue $\lambda(p, q) > 0$ associated to a positive and unique eigenfunction (u^*, v^*) . Further more, this result have been extended by Kandilakis and al. [13] for the system

$$(1.4) \quad \begin{cases} \Delta_p u + \lambda a(x) |u|^{p-2} u + \lambda b(x) |u|^{\alpha-1} |v|^{\beta+1} = 0 & \text{in } \Omega \\ \Delta_q v + \lambda d(x) |u|^{p-2} u + \lambda b(x) |u|^{\alpha+1} |v|^{\beta+1} v = 0 & \text{on } \Omega \\ |\nabla u|^{p-2} \nabla u \cdot \nu + c_1(x) |u|^{p-2} u = 0 & \text{on } \partial\Omega \\ |\nabla v|^{q-2} \nabla v \cdot \nu + c_2(x) |v|^{q-2} v = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω is an unbounded domain in \mathbb{R}^N with non compact and smooth boundary $\partial\Omega$, the constant exponents $0 < \alpha, \beta$ and $1 < p, q < N$ satisfying

$$(1.5) \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1 \text{ and } (\alpha+1) \frac{N-p}{Np} < q, (\beta+1) \frac{N-q}{Nq} < p.$$

Inspired by [3], Khalil and al. in [14] shown that the first eigenvalue $\lambda_{p,q}$ of (1.2) is simple and moreover they established stability (continuity) for the function $(p, q) \mapsto \lambda(p, q)$.

Motivated by the aforementioned papers, in this work we establish the existence of one-parameter family of nontrivial solutions $((\hat{u}_R, \hat{v}_R), \lambda_R^*)$ for all $R > 0$ for problem (1.1). In addition, we show that the corresponding eigenfunction (\hat{u}_R, \hat{v}_R) is positive in Ω , bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$ and belongs to $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$ for certain $\gamma \in (0, 1)$ if $p, q \in C^1(\overline{\Omega}) \cap C^{0,\theta}(\overline{\Omega})$. Furthermore, by means of geometrical conditions on the domain Ω , we prove that the infimum of the eigenvalues of (1.1) is positive. To the best of our knowledge, it is for the first time when the positive infimum eigenvalue for systems involving $p(x)$ -Laplacian operator is studied. However, we point out that in this paper, the existence of an eigenfunction corresponding to the infimum of the eigenvalues of (1.1) is not established and therefore, this issue still remains an open problem.

The rest of the paper is organized as follows. Section 2 contains hypotheses, some auxiliary and useful results involving variable exponent Lebesgue-Sobolev

spaces and our main results. Section 3 and section 4 present the proof of our main results.

2. HYPOTHESES - MAIN RESULTS AND SOME AUXILIARY RESULTS

Let $L^{p(x)}(\Omega)$ be the generalized Lebesgue space that consists of all measurable real-valued functions u satisfying

$$\rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx < +\infty,$$

endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf\{\tau > 0 : \rho_{p(x)}(\frac{u}{\tau}) \leq 1\}.$$

The variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is defined by

$$W_0^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

The norm $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$ makes $W_0^{1,p(x)}(\Omega)$ a Banach space and the following embedding

$$(2.1) \quad W_0^{1,p(x)} \hookrightarrow L^{r(x)}(\Omega)$$

is compact with $1 < r(x) < \frac{Np(x)}{N-p(x)}$.

2.1. Hypotheses.

(H.1): Ω is an bounded open of \mathbb{R}^N , its boundary $\partial\Omega$ of class $C^{2,\delta}$, for certain $0 < \delta < 1$,

(H.2): $c : \Omega \rightarrow \mathbb{R}_+$ and $c \in L^\infty(\Omega)$,

(H.3): $\alpha, \beta : \overline{\Omega} \rightarrow]1, +\infty[$ two continuous functions satisfying

$$1 < \alpha^- = \inf_{x \in \Omega} \alpha(x) \leq \alpha^+ = \sup_{x \in \Omega} \alpha(x) < \infty, 1 < \beta^- = \inf_{x \in \Omega} \beta(x) \leq \beta^+ = \sup_{x \in \Omega} \beta(x) < \infty$$

and

$$\frac{\alpha(x)+1}{p(x)} + \frac{\beta(x)+1}{q(x)} = 1,$$

(H.4): p and q are two variable exponents of class $C^1(\overline{\Omega})$ satisfying

$$p(x) < \frac{Np(x)}{N-p(x)}, q(x) < \frac{Nq(x)}{N-q(x)},$$

with

$$1 < p^- = \inf_{x \in \Omega} p(x) \leq p^+ = \sup_{x \in \Omega} p(x) < \infty, \\ 1 < q^- = \inf_{x \in \Omega} q(x) \leq q^+ = \sup_{x \in \Omega} q(x) < \infty.$$

2.2. Main results. Throughout this paper, we set $X_0^{p(x),q(x)}(\Omega) = W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$.

Define on $X_0^{p(x),q(x)}(\Omega)$ the functionals \mathcal{A} and \mathcal{B} as follows:

$$(2.2) \quad \mathcal{A}(z, w) = \int_{\Omega} \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_{\Omega} \frac{1}{q(x)} |\nabla w|^{q(x)} dx,$$

$$(2.3) \quad \mathcal{B}(z, w) = \int_{\Omega} c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx,$$

and denote by $\|(z, w)\| = \|z\|_{1,p(x)} + \|w\|_{1,q(x)}$. The same reasoning exploited in [8] implies that \mathcal{A} and \mathcal{B} are of class $C^1(X_0^{p(x),q(x)}(\Omega), \mathbb{R})$. The Fréchet derivatives of \mathcal{A} and \mathcal{B} at (z, w) in $X_0^{p(x),q(x)}(\Omega)$ are given by

$$(2.4) \quad \mathcal{A}'(z, w) \cdot (\varphi, \psi) = \int_{\Omega} |\nabla z|^{p(x)-2} \nabla z \cdot \nabla \varphi \, dx + \int_{\Omega} |\nabla w|^{q(x)-2} \nabla w \cdot \nabla \psi \, dx$$

and

$$(2.5) \quad \begin{aligned} \mathcal{B}'(z, w) \cdot (\varphi, \psi) &= \int_{\Omega} c(x)(\alpha(x) + 1)|z|^{\alpha(x)-1}|w|^{\beta(x)+1}\varphi \\ &\quad + \int_{\Omega} c(x)(\beta(x) + 1)|z|^{\alpha(x)+1}|w|^{\beta(x)-1}w\psi \, dx, \end{aligned}$$

where $(\varphi, \psi) \in X_0^{p(x),q(x)}(\Omega)$.

Let $R > 0$ be fixed, we set

$$\mathcal{X}_R = \{(z, w) \in X_0^{p(x),q(x)}(\Omega); \mathcal{B}(z, w) = R\}.$$

It is obvious to notice that the set \mathcal{X}_R is not empty. Indeed, let $(z_0, w_0) \in X_0^{p(x),q(x)}(\Omega)$ such that $\mathcal{B}(z_0, w_0) = b_0 > 0$, if $b_0 = R$, we done. Otherwise, for $z_R = (R/b_0)^{1/p(x)}z_0$ and $w_R = (R/b_0)^{1/q(x)}w_0$, it is easy to note that $\mathcal{B}(z_R, w_R) = R$.

Now, define the Rayleigh quotients

$$(2.6) \quad \lambda_R^* = \inf_{(z,w) \in \mathcal{X}_R} \frac{\mathcal{A}(z,w)}{\mathcal{B}(z,w)},$$

$$\lambda_{p(x),q(x)}^* = \inf_{(z,w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}} \frac{\mathcal{A}(z,w)}{\mathcal{B}(z,w)}$$

and

$$(2.7) \quad \lambda_{*R} = \inf_{(z,w) \in \mathcal{X}_R} \frac{\mathcal{A}(z,w)}{\int_{\Omega} c(x)(\alpha(x)+\beta(x)+2)|z|^{\alpha(x)+1}|w|^{\beta(x)+1}dx}.$$

Remark 1. The constant λ_R^* in (2.6) can be written as follows

$$(2.8) \quad R\lambda_R^* = \inf_{\{(z,w) \in \mathcal{X}_R\}} \mathcal{A}(z, w).$$

Our first main result provides the existence of a one - parameter family of solutions for the system (1.1).

Theorem 1. Assume that (H.1) - (H.4) hold. Then, the system (1.1) has a one-parameter family of nontrivial solutions $((\hat{u}_R, \hat{v}_R), \lambda_R^*)$ for all $R \in (0, +\infty)$. Moreover, if one of the following conditions holds:

(a.1): There is vectors $l_1, l_2 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$, $f(t_1) = p(x + t_1 l_1)$ and $g(t_2) = q(x + t_2 l_2)$ are monotone for $t_i \in I_{i,x} = \{t_i; x + t_i l_i \in \Omega\}$, $i = 1, 2$.

(a.2): There is $x_1, x_2 \notin \overline{\Omega}$ such that for all $w_1, w_2 \in \mathbb{R} \setminus \{0\}$ with $\|w_1\|, \|w_2\| = 1$, the functions $f(t_1) = p(x_0 + t_1 w_1)$ and $g(t_2) = p(x_2 + t_2 w_2)$ are monotone for $t_i \in I_{x_i, w_i} = \{t_i \in \mathbb{R}; x_i + t_i w_i \in \Omega\}$, $i = 1, 2$.

Then, $\lambda_{p(x),q(x)}^* = \inf_{R>0} \lambda_R^* > 0$ is the positive infimum eigenvalue of problem (1.1).

A second main result consists in positivity, boundedness and regularity for the obtained solution of problem (1.1).

Theorem 2. Let R be a fixed and strictly positive real. Assume that (H.3) holds. Then, (\hat{u}_R, \hat{v}_R) the nontrivial solution of problem (1.1) is positive and bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$. Moreover, if $p, q \in C^1(\overline{\Omega}) \cap C^{0,\gamma}(\overline{\Omega})$ for certain $\gamma \in (0, 1)$ then (\hat{u}_R, \hat{v}_R) belongs to $C^{1,\delta}(\overline{\Omega}) \times C^{1,\delta}(\overline{\Omega})$, $\delta \in (0, 1)$.

The proof of Theorem 1 will be done in section 3 while in section 4 we will present the proof of Theorem 2.

2.3. Some preliminaries lemmas.

Lemma 1. (i) For any $u \in L^{p(x)}(\Omega)$ we have

$$\|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+} \quad \text{if } \|u\|_{p(x)} > 1,$$

$$\|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-} \quad \text{if } \|u\|_{p(x)} \leq 1.$$

(ii) For $u \in L^{p(x)}(\Omega) \setminus \{0\}$ we have

$$(2.9) \quad \|u\|_{p(x)} = a \quad \text{if and only if } \rho_{p(x)}\left(\frac{u}{a}\right) = 1.$$

Lemma 2 ([4, Theorem 8.2.4]). For every $u \in W_0^{1,p(\cdot)}(\Omega)$ the inequality

$$(2.10) \quad \|u\|_{L^{p(\cdot)}(\Omega)} \leq C_{N,p} \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

holds with a constant $C_{N,p} > 0$.

Recall that if there exist a constant $L > 0$ and an exponent $\theta \in (0, 1)$ such that

$$|p(x_1) - p(x_2)| \leq L|x_1 - x_2|^\theta \quad \text{for all } x_1, x_2 \in \overline{\Omega},$$

then the function p is said to be Hölder continuous on $\overline{\Omega}$ and we denote $p \in C^{0,\theta}(\overline{\Omega})$.

For a later use, we have the next result.

Lemma 3. For $s \in (0, 1)$ it holds

$$\sum_{n=1}^r (n-1)s^{n-1} \leq \frac{s}{(s-1)^2}.$$

Proof. Recall that for $s > 0$ we have

$$s^r - 1 = (s-1)(s^{r-1} + s^{r-2} + \dots + s + 1), \quad \forall r \in \mathbb{N}^*.$$

Multiplying by s one get

$$\begin{aligned} rs^r &= (s^r + s^{r-1} + \dots + s) + (s-1)((r-1)s^{r-1} + \dots + s) \\ &= \frac{s-s^{r+1}}{1-s} + (s-1)((r-1)s^{r-1} + \dots + s), \end{aligned}$$

for all $s \neq 1$. Thus, it follows that

$$(r-1)s^{r-1} + \dots + s = \frac{s-s^{r+1}}{(s-1)^2} - \frac{rs^{r-1}}{1-s}.$$

Hence, for $0 < s < 1$ one has

$$(r-1)s^{r-1} + \dots + s \leq \frac{s}{(s-1)^2}.$$

■

3. PROOF OF THEOREM 1

Taking account of the assumption (H.3), we note that the system (1.1) is arising from a nonlinear eigenvalue type problem. Solvability of general class of nonlinear eigenvalues problems of type $\mathcal{A}'(x) = \lambda \mathcal{B}'(x)$ have been treated by M.S Berger in [1]. We recall this main tool.

Theorem 3. [1] *Suppose that the C^1 functionals \mathcal{A} and \mathcal{B} defined on the reflexive Banach space X have the following properties:*

- (1) \mathcal{A} is weakly lower semicontinuous and coercive on $X \cap \{\mathcal{B}(x) \leq \text{const.}\}$;
- (2) \mathcal{B} is continuous with respect to weak sequential convergence and $\mathcal{B}'(x) = 0$ only at $x = 0$.

Then the equation $\mathcal{A}'(x) = \lambda \mathcal{B}'(x)$ has a one-parameter family of nontrivial solutions (x_R, λ_R) for all R in the range of $\mathcal{B}(x)$ such that $\mathcal{B}(x_R) = R$; and x_R is characterized as the minimum of $\mathcal{A}(x)$ over the set $\{\mathcal{B}(x) = R\}$.

Remark 2. *In the statement (ii) of the theorem 3, the condition “ $\mathcal{B}'(x) = 0$ only at $x = 0$ ” may be replaced by “ $\mathcal{B}(x) = 0$ only at $x = 0$ ”. Indeed, in the proof of Theorem 3, assume that the minimizing problem $\inf_{\{\mathcal{B}(x)=R\}} \mathcal{A}(x)$ is attained at $x_R \in X$ then because \mathcal{A} and \mathcal{B} are differentiable there exists (λ_1, λ_2) a pair of Lagrange multipliers such that*

$$\lambda_1 \mathcal{A}'(x_R) + \lambda_2 \mathcal{B}'(x_R) = 0.$$

Consequently, λ_1 and λ_2 are not both zero. In fact, if $\lambda_2 \neq 0$ and $\lambda_1 = 0$ then we get

$$\lambda_2 (\mathcal{B}'(x_R), x_R) = 0.$$

So, for instance, assume that the following condition obeys “there exists $\gamma > 0$ such that

$$(\mathcal{B}'(x), x) \geq \gamma \mathcal{B}(x) \text{ for all } x \in X.$$

In this case, particularly, taking $x = x_R$, it follows that $(\mathcal{B}'(x_R), x_R) = 0$ implies $\mathcal{B}(x_R) = 0$. This is a contradiction because x_R belongs in the set $\{\mathcal{B}(x) = R\}$.

3.1. Properties on \mathcal{A} and \mathcal{B} .

Lemma 4. (i) $\mathcal{A}(z, w)$ is coercive on $X_0^{p(x), q(x)}(\Omega)$.

(ii) \mathcal{B} is a weakly continuous functional, namely, $(z_n, w_n) \rightharpoonup (z, w)$ (weak convergence) implies $\mathcal{B}(z_n, w_n) \rightarrow \mathcal{B}(z, w)$.

(iii) Let (z, w) be in $X_0^{p(x), q(x)}(\Omega)$. Assume $\mathcal{B}'(z, w) = 0$ in $X^{-1, p'(x), q'(x)}(\Omega)$ then $\mathcal{B}(z, w) = 0$.

Proof. (i) For any $(z, w) \in X_0^{p(x), q(x)}(\Omega)$ with $\|z\|_{1, p(x)}, \|w\|_{1, q(x)} > 1$, using Lemma 1 we have

$$\begin{aligned} & \int_{\Omega} \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_{\Omega} \frac{1}{q(x)} |\nabla w|^{q(x)} dx \\ & \geq \frac{1}{p^+} \int_{\Omega} |\nabla z|^{p(x)} dx + \frac{1}{q^+} \int_{\Omega} |\nabla w|^{q(x)} dx \\ & \geq \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} (\|z\|_{1, p(x)}^{p^-} + \|w\|_{1, q(x)}^{q^-}) \\ & \geq 2^{-\min\{p^-, q^-\}} \min\left\{\frac{1}{p^+}, \frac{1}{q^+}\right\} (\|z\|_{1, p(x)} + \|w\|_{1, q(x)})^{\min\{p^-, q^-\}}. \end{aligned}$$

Since $\min\{p^-, q^-\} > 1$ (see (H.3) and (H.4)) the above inequality implies that

$$\mathcal{A}(z, w) \rightarrow \infty \text{ as } \|(z, w)\| \rightarrow \infty.$$

(ii) Let $(z_n, w_n) \rightharpoonup (z, w)$ in $X_0^{p(x), q(x)}(\Omega)$. By the first part in (H.4) and (2.1) the embeddings $W_0^{1, p(x)} \hookrightarrow L^{p(x)}(\Omega)$ and $W_0^{1, q(x)} \hookrightarrow L^{q(x)}(\Omega)$ are both compact, so we get

$$(3.1) \quad (z_n, w_n) \rightarrow (z, w) \text{ in } L^{p(x)}(\Omega) \times L^{q(x)}(\Omega).$$

Using (H.3) and the definition of \mathcal{B} , we have

$$\begin{aligned} |\mathcal{B}(z_n, w_n) - \mathcal{B}(z, w)| &\leq \|c\|_\infty \left[\int_\Omega |z|^{\alpha(x)+1} (|w|^{\beta(x)+1} - |w_n|^{\beta(x)+1}) dx \right. \\ &\quad \left. + \int_\Omega |w_n|^{\alpha(x)+1} (|z|^{\alpha(x)+1} - |z_n|^{\alpha(x)+1}) dx \right] \\ &\leq 2^{\max(\alpha^+, \beta^+)} \|c\|_\infty \left[\int_\Omega |z|^{\alpha(x)+1} |w - w_n|^{\beta(x)+1} dx \right. \\ &\quad \left. + \int_\Omega |w_n|^{\alpha(x)+1} |z - z_n|^{\alpha(x)+1} dx \right]. \end{aligned}$$

By Hölder inequality one has

$$\begin{aligned} &\int_\Omega |z|^{\alpha(x)+1} |w - w_n|^{\beta(x)+1} dx \\ &\leq C_{\alpha, \beta, p, q} \| |z|^{\alpha(x)+1} \|_{L^{\frac{p(x)}{\alpha(x)+1}}(\Omega)} \| |w - w_n|^{\alpha(x)+1} \|_{L^{\frac{q(x)}{\beta(x)+1}}(\Omega)}. \end{aligned}$$

where $C_{\alpha, \beta, p, q} > 0$ is a constant. Observe that

$$\| |w - w_n|^{\beta(x)+1} \|_{L^{\frac{q(x)}{\beta(x)+1}}(\Omega)}^{q^+} \leq \int_\Omega (|w - w_n|^{\beta(x)+1})^{\frac{q(x)}{\beta(x)+1}} dx = \rho_{q(\cdot)}(w - w_n)$$

and

$$\rho_{q(\cdot)}(w - w_n) \leq \| |w - w_n|^{\beta(x)+1} \|_{L^{q(x)}(\Omega)}^{q^-}.$$

Then it follows that

$$\| |w - w_n|^{\beta(x)+1} \|_{L^{\frac{q(x)}{\beta(x)+1}}(\Omega)}^{q^+} \leq \rho_{q(\cdot)}(w - w_n)^{1/q^+} \leq \| |w - w_n|^{\beta(x)+1} \|_{L^{q(x)}(\Omega)}^{q^-/q^+}.$$

Therefore, the strong convergence in (3.1) ensures that

$$\| |w - w_n|^{\beta(x)+1} \|_{L^{\frac{q(x)}{\beta(x)+1}}(\Omega)} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

A quite similar argument provides

$$\| |z - z_n|^{\alpha(x)+1} \|_{L^{\frac{p(x)}{\alpha(x)+1}}(\Omega)} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

(iii) From (2.5), it is clear to notice that for any $(z, w) \in X_0^{p(x), q(x)}(\Omega)$, by taking $\varphi = 1/p(x)z$ and $\psi = 1/q(x)w$, the following identity holds

$$\mathcal{B}'(z, w), (1/p(x)z, 1/q(x)w) = \mathcal{B}(z, w).$$

Then the statement (iii) follows. This conclude the proof of the Lemma. ■

3.2. A priori bound for \mathcal{A} .

Lemma 5. *Let R a fixed and strictly positive real. There exists a constant $\mathcal{K}(R) > 0$ depending on R such that*

$$(3.2) \quad \mathcal{A}(z, w) \geq \mathcal{K}(R) > 0, \quad \forall (z, w) \in \mathcal{X}_R.$$

Proof. First, observe from Lemma 2 that if $\|\nabla z\|_{L^{p(x)}(\Omega)} < 1$, we have

$$\left\| \frac{z}{C_{N, p}} \right\|_{L^{p(x)}(\Omega)} < 1.$$

Then it follows that

$$(3.3) \quad \rho_{p(x)}\left(\frac{z}{C_{N,p}}\right) \leq \left\| \frac{z}{C_{N,p}} \right\|_{L^{p(x)}(\Omega)}^{p^-},$$

which combined with Lemma 2 leads to

$$\int_{\Omega} \frac{|z|^{p(x)}}{C_{N,p}^{p(x)}} dx \leq \|\nabla z\|_{L^{p(x)}(\Omega)}^{p^-}.$$

Hence it holds

$$(3.4) \quad \int_{\Omega} |z|^{p(x)} dx \leq K_{N,p} \|\nabla z\|_{L^{p(x)}(\Omega)}^{p^-} \leq K_{N,p} \|\nabla z\|_{L^{p(x)}(\Omega)}^{p^-/p^+},$$

where

$$K_{N,p} = \begin{cases} C_{N,p}^{p^+} & \text{if } C_{N,p} > 1 \\ C_{N,p}^{p^-} & \text{if } C_{N,p} < 1. \end{cases}$$

A quite similar argument shows that

$$(3.5) \quad \int_{\Omega} |w|^{q(x)} dx \leq K_{N,q} \|\nabla w\|_{L^{q(x)}(\Omega)}^{q^-/q^+},$$

where

$$K_{N,q} = \begin{cases} C_{N,q}^{q^+} & \text{if } C_{N,q} > 1 \\ C_{N,q}^{q^-} & \text{if } C_{N,q} < 1. \end{cases}$$

For every $(z, w) \in X_0^{p(x), q(x)}(\Omega)$, Young inequality and (H.3) imply

$$(3.6) \quad \begin{aligned} \int_{\Omega} c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx &\leq \|c\|_{\infty} \int_{\Omega} \left[\frac{\alpha(x)+1}{p(x)} |z|^{p(x)} + \frac{\beta(x)+1}{q(x)} |w|^{q(x)} \right] dx \\ &\leq \|c\|_{\infty} \left(\int_{\Omega} |z|^{p(x)} dx + \int_{\Omega} |w|^{q(x)} dx \right). \end{aligned}$$

Assume that $(z, w) \in \mathcal{X}_R$ is such that

$$(3.7) \quad \max \left(\|\nabla z\|_{L^{p(\cdot)}(\Omega)}, \|\nabla w\|_{L^{q(\cdot)}(\Omega)} \right) < 1.$$

Bearing in mind (H.3), (H.4) and (i) of Lemma 1, we have

$$(3.8) \quad \max \left\{ \int_{\Omega} \frac{1}{p(x)} |\nabla z|^{p(x)} dx, \int_{\Omega} \frac{1}{q(x)} |\nabla w|^{q(x)} dx \right\} < 1.$$

Then, from (3.4)-(3.8), it follows that

$$(3.9) \quad R \leq K_1 \left(\int_{\Omega} \frac{1}{p(x)} |\nabla z|^{p(x)} dx \right)^{p^-/p^+} + K_2 \left(\int_{\Omega} \frac{1}{q(x)} |\nabla w|^{q(x)} dx \right)^{q^-/q^+}$$

From the hypothesis (H.4) on p^- , p^+ , q^- and q^+ , it follows that

$$(3.10) \quad \begin{aligned} R^{\frac{p^+q^+}{p^-q^-}} &\leq 2^{\frac{p^+q^+}{p^-q^-}-1} \left[K_1^{\frac{p^+q^+}{p^-q^-}} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla z|^{p(x)} dx \right)^{q^+/q^-} \right. \\ &\quad \left. + K_2^{p^+q^+/p^-q^-} \left(\int_{\Omega} \frac{1}{q(x)} |\nabla w|^{q(x)} dx \right)^{p^+/p^-} \right]. \end{aligned}$$

Or again

$$(3.11) \quad R^{\frac{p^+q^+}{p^-q^-}} \leq (2K_3)^{\frac{p^+q^+}{p^-q^-}} \left[\int_{\Omega} \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_{\Omega} \frac{1}{q(x)} |\nabla w|^{q(x)} dx \right]$$

where

$$K_1 = K_{N,p} (p^+)^{p^-/p^+} \|c\|_{\infty}, \quad K_2 = K_{N,q} (q^+)^{q^-/q^+} \|c\|_{\infty}$$

and $K_3 = K_1 + K_2$. Thus, from (3.11), we conclude that

$$(3.12) \quad \mathcal{A}(z, w) \geq \left(\frac{R}{2K_3}\right)^{\frac{q^+ p^+}{q^- p^-}}.$$

Now, we deal with the case when $(z, w) \in \mathcal{X}_R$ is such that

$$\max(\|\nabla z\|_{L^{p(\cdot)}(\Omega)}, \|\nabla w\|_{L^{q(\cdot)}(\Omega)}) \geq 1.$$

This implies that

$$\max\left(\int_{\Omega} |\nabla z|^{p(x)} dx, \int_{\Omega} |\nabla w|^{q(x)} dx\right) \geq 1.$$

If $\int_{\Omega} |\nabla z|^{p(x)} dx \geq 1$ we have

$$p^+ \int_{\Omega} \frac{1}{p(x)} |\nabla z|^{p(x)} dx \geq \int_{\Omega} |\nabla z|^{p(x)} dx \geq 1.$$

which in turn yields

$$(3.13) \quad \mathcal{A}(z, w) = \int_{\Omega} \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_{\Omega} \frac{1}{q(x)} |\nabla w|^{q(x)} dx > \frac{1}{p^+}.$$

Now for $\int_{\Omega} |\nabla w|^{q(x)} dx \geq 1$ a quite similar argument provides

$$(3.14) \quad \mathcal{A}(z, w) > \frac{1}{q^+}.$$

We notice that if $\max(\|\nabla z\|_{L^{p(\cdot)}(\Omega)}, \|\nabla w\|_{L^{q(\cdot)}(\Omega)}) \geq 1$, from (3.13) and (3.14), it is clearly that

$$(3.15) \quad \mathcal{A}(z, w) > \max\left(\frac{1}{p^+}, \frac{1}{q^+}\right).$$

Thus, according to (3.12) and (3.15), for all $(z, w) \in \mathcal{X}_R$, one has

$$(3.16) \quad \mathcal{A}(z, w) \geq \max\left\{\left(\frac{R}{2K_3}\right)^{\frac{q^+ p^+}{q^- p^-}}, \frac{1}{p^+}, \frac{1}{q^+}\right\} > 0.$$

Consequently, there exists a constant $\mathcal{K}(R) > 0$ depending on R such that (3.2) holds. ■

3.3. Proof of (2.8). We begin by the proposition.

Proposition 1. *Assume that (H.3) holds. Then, for $R > 0$,*

- (i): $0 < \frac{\lambda_R^*}{(\alpha^+ + \beta^+ + 2)} < \lambda_{*R} < \lambda_R^*$.
- (ii): *Any $\lambda < \lambda_{*R}$ is not an eigenvalue of problem (1.1).*
- (iii): *There exists $(\hat{u}_R, \hat{v}_R) \in \mathcal{X}_R$ such that λ_R^* is a corresponding eigenvalue for the system (1.1).*

Proof. (i). First let us show that $0 < \frac{\lambda_R^*}{(\alpha^+ + \beta^+ + 2)} \leq \lambda_{*R} \leq \lambda_R^*$. Obviously, for all $(z, w) \in \mathcal{X}_R$, we have

$$\frac{\mathcal{A}(z, w)}{(\alpha^+ + \beta^+ + 2)R} \leq \frac{\mathcal{A}(z, w)}{\int_{\Omega} c(x)(\alpha(x) + \beta(x) + 2)|z|^{\alpha(x)+1}|w|^{\beta(x)+1} dx} \leq \frac{\mathcal{A}(z, w)}{R}.$$

from (2.6) and (2.7), it derives that $\frac{\lambda_R^*}{(\alpha^+ + \beta^+ + 2)} < \lambda_{*R} < \lambda_R^*$. Now suppose that $\lambda_{*R} = 0$. Then $\lambda_R^* = 0$ and by virtue of Lemma 5 and Remark 1 this is a contradiction. Hence $\lambda_{*R} > 0$.

(ii). Next we show that λ cannot be an eigenvalue for $\lambda < \lambda_{*}$. Indeed, suppose by contradiction that λ is an eigenvalue of problem (1.1). Then there exists $(u, v) \in X_0^{p(x), q(x)}(\Omega) - \{(0, 0)\}$ such that

$$(3.17) \quad \begin{aligned} \int_{\Omega} |\nabla u|^{p(x)} dx &= \lambda \int_{\Omega} c(x)(\alpha(x) + 1)|u|^{\alpha(x)+1}|v|^{\beta(x)+1} \\ \int_{\Omega} |\nabla v|^{q(x)} dx &= \lambda \int_{\Omega} c(x)(\beta(x) + 1)|u|^{\alpha(x)+1}|v|^{\beta(x)+1}. \end{aligned}$$

On the basis of (H.3), (H.4), (2.7) and (3.17), we get

$$\begin{aligned}
& \lambda_* \int_{\Omega} c(x)(\alpha(x) + \beta(x) + 2)|u|^{\alpha(x)+1}|v|^{\beta(x)+1} dx \\
& \leq \int_{\Omega} \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} \right) dx \\
& \leq \int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\Omega} |\nabla v|^{q(x)} dx \\
& = \lambda \int_{\Omega} c(x)(\alpha(x) + \beta(x) + 2)|u|^{\alpha(x)+1}|v|^{\beta(x)+1} dx \\
& < \lambda_* \int_{\Omega} c(x)(\alpha(x) + \beta(x) + 2)|u|^{\alpha(x)+1}|v|^{\beta(x)+1} dx,
\end{aligned}$$

which is not possible and the conclusion follows.

(iii). Now, we claim that the infimum in (2.8) is achieved at an element of \mathcal{X}_R . Indeed, thanks to the lemma 4, \mathcal{B} is weakly continuous on $X_0^{p(x),q(x)}(\Omega)$, then the nonempty set \mathcal{X}_R is weakly closed. So, since \mathcal{A} is weakly lower semicontinuous, we conclude that there exists an element of \mathcal{X}_R which we denote (\hat{u}, \hat{v}_R) such that (2.8) is feasible. Since $(\hat{u}_R, \hat{v}_R) \neq 0$, we also have $\mathcal{B}'(\hat{u}_R, \hat{v}_R) \neq 0$ otherwise it implies $\mathcal{B}(\hat{u}_R, \hat{v}_R) = 0$ and which contradicts $(\hat{u}, \hat{v}_R) \in \mathcal{X}_R$. So, owing to Lagrange multiplier method (see e.g. [1, Theorem 6.3.2, p. 325] or [5, Theorem 6.3.2, p. 402]), there exists $\lambda_R \in \mathbb{R}$ such that

$$(3.18) \quad \mathcal{A}'(\hat{u}_R, \hat{v}_R) \cdot (\varphi, \psi) = \lambda_R \mathcal{B}'(\hat{u}_R, \hat{v}_R) \cdot (\varphi, \psi), \quad \forall (\varphi, \psi) \in X_0^{p(x),q(x)}(\Omega)$$

where \mathcal{A}' and \mathcal{B}' are defined as in (2.4) and (2.5) respectively.

In the sequel, we show that λ_R is equal to λ_R^* . To this end, let us denote by Ω^+ and Ω^- the sets defined as follows

$$\Omega^+ = \{x \in \Omega; |\nabla \hat{u}_R|^{p(x)} - \lambda_R(\alpha(x) + 1)c(x)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1} \geq 0\}$$

and

$$\Omega_- = \{x \in \Omega; |\nabla \hat{u}_R|^{p(x)} - \lambda_R(\alpha(x) + 1)c(x)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1} < 0\}.$$

By taking $\varphi = \hat{u}_R \mathbf{1}_{\Omega^+}$ and $\psi = 0$ in (3.18) one has

$$(3.19) \quad \int_{\Omega^+} (|\nabla \hat{u}_R|^{p(x)} - \lambda_R c(x)(\alpha(x) + 1)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1}) dx = 0$$

and likewise, by choosing $\varphi = \hat{u}_R \mathbf{1}_{\Omega^-}$ and $\psi = 0$ in (3.18) we get

$$(3.20) \quad \int_{\Omega^-} (|\nabla \hat{u}_R|^{p(x)} - \lambda_R c(x)(\alpha(x) + 1)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1}) dx = 0.$$

We claim that

$$(3.21) \quad \int_{\Omega} \frac{1}{p(x)} |\nabla \hat{u}_R|^{p(x)} dx = \lambda_R \int_{\Omega} c(x) \frac{\alpha(x)+1}{p(x)} |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} dx.$$

Indeed, on account of (H.4), (3.19) and (3.20) we have

$$\begin{aligned}
& \left| \int_{\Omega} \frac{|\nabla \hat{u}_R|^{p(x)}}{p(x)} dx - \lambda_R \int_{\Omega} \frac{\alpha(x)+1}{p(x)} c(x) |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} dx \right| \\
& \leq \int_{\Omega} p(x) \left| \frac{|\nabla \hat{u}_R|^{p(x)}}{p(x)} - \lambda_R \frac{\alpha(x)+1}{p(x)} c(x) |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} \right| dx \\
& = \int_{\Omega} \left| |\nabla \hat{u}_R|^{p(x)} - \lambda_R(\alpha(x) + 1)c(x)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1} \right| dx \\
& \leq \int_{\Omega^+} (|\nabla \hat{u}_R|^{p(x)} - \lambda_R(\alpha(x) + 1)c(x)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1}) dx \\
& \quad - \int_{\Omega^-} (|\nabla \hat{u}_R|^{p(x)} - \lambda_R(\alpha(x) + 1)c(x)|\hat{u}_R|^{\alpha(x)+1}|\hat{v}_R|^{\beta(x)+1}) dx = 0,
\end{aligned}$$

showing that (3.21) holds. In the same manner we can prove that

$$(3.22) \quad \int_{\Omega} \frac{1}{q(x)} |\nabla \hat{v}_R|^{q(x)} dx = \lambda_R \int_{\Omega} c(x) \frac{\beta(x)+1}{q(x)} |\hat{u}_R|^{\alpha(x)+1} |\hat{v}_R|^{\beta(x)+1} dx.$$

Adding together (3.21) and (3.22), on account of (H.3) and (3.14), we achieve that

$$\mathcal{A}(\hat{u}_R, \hat{v}_R) = R\lambda_R.$$

Then, bearing in mind (3.15) it turns out that $\lambda_R = \lambda_R^*$, showing that λ_R^* is at least one eigenvalue of (1.1).

Then, combining this last point with the characterization (3.18), we get

$$\mathcal{A}'(\hat{u}_R, \hat{v}_R) \cdot (\varphi, 0) = \lambda_R^* \mathcal{B}'(\hat{u}_R, \hat{v}_R) \cdot (\varphi, 0), \quad \forall \varphi \in W_0^{1,q(x)}(\Omega)$$

and

$$\mathcal{A}'(\hat{u}_R, \hat{v}_R) \cdot (0, \psi) = \lambda_R^* \mathcal{B}'(\hat{u}_R, \hat{v}_R) \cdot (0, \psi), \quad \forall \psi \in W_0^{1,q(x)}(\Omega).$$

On other words, it means that $((\hat{u}_R, \hat{v}_R), \lambda_R^*)$ is a solution of the system (1.1). ■

3.4. Proof of Theorem 1. Employing again the statement of Lemma 4, we can apply the theorem 3 due to [1]. Then the system (1.1) has a one-parameter family of nontrivial solutions $((\hat{u}_R, \hat{v}_R), \lambda_R)$ for all $R > 0$. Moreover, from (iii) of Proposition 1, $\lambda_R = \lambda_R^*$.

It remains to prove that $\lambda_{p(x),q(x)}^* = \inf_{R>0} \lambda_R^* > 0$. From (3.6) and for $(z, w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}$, one has

$$(3.23) \quad \frac{1}{\|c\|_\infty} \cdot \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega |z|^{p(x)} dx + \int_\Omega |w|^{q(x)} dx} \leq \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx}.$$

Recalling that under assumption (a.1) or (a.2), the authors in [9] proved that the first eigenvalues

$$(3.24) \quad \begin{cases} \lambda_{p(x)}^* = \inf_{z \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla z|^{p(x)} dx}{\int_\Omega |z|^{p(x)} dx} \\ \lambda_{q(x)}^* = \inf_{z \in W_0^{1,q(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla z|^{q(x)} dx}{\int_\Omega |z|^{q(x)} dx}, \end{cases}$$

are strictly positive. Hence, combining with (3.23) it follows that

$$\begin{aligned} \min \left\{ \frac{\lambda_{p(x)}^*}{p^+ \|c\|_\infty}, \frac{\lambda_{q(x)}^*}{q^+ \|c\|_\infty} \right\} &= \min \left\{ \frac{\lambda_{p(x)}^*}{p^+ \|c\|_\infty}, \frac{\lambda_{q(x)}^*}{q^+ \|c\|_\infty} \right\} \cdot \frac{\int_\Omega |z|^{p(x)} dx + \int_\Omega |w|^{q(x)} dx}{\int_\Omega |z|^{p(x)} dx + \int_\Omega |w|^{q(x)} dx} \\ &\leq \frac{\frac{\lambda_{p(x)}^*}{p^+ \|c\|_\infty} \int_\Omega |z|^{p(x)} dx + \frac{\lambda_{q(x)}^*}{q^+ \|c\|_\infty} \int_\Omega |w|^{q(x)} dx}{\int_\Omega |z|^{p(x)} dx + \int_\Omega |w|^{q(x)} dx} \leq \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx}. \end{aligned}$$

Then

$$(3.25) \quad \begin{aligned} 0 &< \min \left\{ \frac{\lambda_{p(x)}^*}{p^+ \|c\|_\infty}, \frac{\lambda_{q(x)}^*}{q^+ \|c\|_\infty} \right\} \\ &\leq \inf_{(z,w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx}. \end{aligned}$$

Another hand, since

$$\bigcup_{R>0} \mathcal{X}_R \subset \left\{ (z, w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\} \right\},$$

one gets

$$(3.26) \quad \begin{aligned} \inf_{(z,w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx} \\ \leq \inf_{\{\mathcal{B}(z,w)=R\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx}. \end{aligned}$$

Thus, gathering (3.25) and (3.26) together we infer that

$$0 < \min \left\{ \frac{\lambda_{p(x)}^*}{p^+ \|c\|_\infty}, \frac{\lambda_{q(x)}^*}{q^+ \|c\|_\infty} \right\} \leq \lambda_{p(x),q(x)}^* \leq \inf_{R>0} \lambda_R^*.$$

Next, let us prove that $\lambda_{p(x),q(x)}^* \geq \inf_{R>0} \lambda_R^*$. To this end, let a constant $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that $\lambda_{R_\varepsilon}^* < \inf_{R>0} \lambda_R^* + \varepsilon$. This implies that

$$(3.27) \quad \lambda_{R_\varepsilon}^* < \lambda_R^* + \varepsilon \text{ for all } R > 0 \text{ and } \varepsilon > 0.$$

Now, let $(z, w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}$ such that $\mathcal{B}(z, w) > 0$ and assume that $R_{(z,w)} = \mathcal{B}(z, w)$. According to (iii) in Proposition 1, the constant

$$\lambda_{R_{(z,w)}}^* = \inf_{\{\mathcal{B}(z,w)=R_{(z,w)}\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx}$$

exists and then

$$\lambda_{R_{(z,w)}}^* \leq \frac{\int_\Omega \frac{1}{p(x)} |\nabla z|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla w|^{q(x)} dx}{\int_\Omega c(x) |z|^{\alpha(x)+1} |w|^{\beta(x)+1} dx}.$$

At this point, combining with (3.27) yields

$$\lambda_{R_\varepsilon}^* < \lambda_{R_{(z,w)}}^* + \varepsilon \leq \frac{A(z,w)}{\mathcal{B}(z,w)} + \varepsilon \text{ for all } \varepsilon > 0,$$

which, it turn, leads to

$$\lambda_{R_\varepsilon}^* < \lambda_{R_{(z,w)}}^* + \varepsilon \leq \inf_{(z,w) \in X_0^{1,p(x),q(x)}(\Omega) \setminus \{0\}} \frac{A(z,w)}{\mathcal{B}(z,w)} + \varepsilon \text{ for all } \varepsilon > 0.$$

This is equivalent to $\lambda_{R_\varepsilon}^* \leq \lambda_{p(x),q(x)}^* + \varepsilon$. Consequently,

$$\inf_{R>0} \lambda_R^* \leq \lambda_{R_\varepsilon}^* \leq \lambda_{p(x),q(x)}^* + \varepsilon \leq \inf_{R>0} \lambda_R^* + \varepsilon \text{ for all } \varepsilon > 0.$$

Finally, passing to the limit as $\varepsilon \rightarrow 0$ implies that $\lambda_{p(x),q(x)}^* = \inf_{R>0} \lambda_R^*$. This ends the proof of Theorem 1.

4. PROOF OF THEOREM 2

Let $(\hat{u}_R, \hat{v}_R) \in X_0^{p(x),q(x)}(\Omega)$ be a solution of problem (1.1) corresponding to the positive infimum eigenvalue λ_R^* and let $d > 0$ be a constant such that

$$(4.1) \quad d = \frac{\hat{d}}{\max\{p^+, q^+\}},$$

where

$$(4.2) \quad 1 < \max\{p^+, q^+\} < \hat{d} \leq \max\{p^+, q^+\} \cdot \min\left\{\frac{\pi_p^-}{p^-}, \frac{\pi_p^+}{p^+}, \frac{\pi_q^-}{q^-}, \frac{\pi_q^+}{q^+}\right\}$$

and

$$(4.3) \quad \pi_p(x) = \frac{Np(x)}{N-p(x)}, \quad \pi_p^- = \inf_{x \in \Omega} \pi_p(x) \quad \text{and} \quad \pi_p^+ = \sup_{x \in \Omega} \pi_p(x).$$

In this section, the goal consists in proving that (\hat{u}_R, \hat{v}_R) is bounded in Ω . Notice that from the above section, we have

$$(4.4) \quad \begin{cases} \int_\Omega |\nabla \hat{u}_R|^{p(x)-2} \nabla \hat{u}_R \nabla \varphi dx = \lambda_R^* \int_\Omega c(x) (\alpha(x) + 1) \hat{u}_R |\hat{u}_R|^{\alpha(x)-1} |\hat{v}_R|^{\beta(x)+1} \varphi dx \\ \int_\Omega |\nabla \hat{v}_R|^{q(x)-2} \nabla \hat{v}_R \nabla \psi dx = \lambda_R^* \int_\Omega c(x) (\beta(x) + 1) |\hat{u}_R|^{\alpha(x)+1} \hat{v}_R |\hat{v}_R|^{\beta(x)-1} \psi dx. \end{cases}$$

Remark 3. From the density of $C_c^\infty(\Omega)$ in $W_0^{1,p(x)}(\Omega)$ and through the embeddings $C_c^\infty(\Omega) \subset C^1(\overline{\Omega})$, $C^1(\overline{\Omega}) \subset W_0^{1,p^+}(\Omega)$ and $W_0^{1,p^+}(\Omega) \subset W_0^{1,p(x)}(\Omega)$ (since $p(x) \leq p^+$ in Ω), we may assume that $\hat{u}_R \in C^1(\overline{\Omega})$ (see, e.g., [7]). The same argument enable us to assume that $\hat{v}_R \in C^1(\overline{\Omega})$.

For a better reading, we divide the proof of Theorem 2 in several lemmas.

Lemma 6. *Assume hypotheses (H.1)-(H.4) hold. Then, for any fixed k in \mathbb{N} , there exist $x_k, y_k \in \Omega$ such that the following estimates hold:*

$$(4.5) \quad \int_{\Omega} \hat{u}_R^{1+p(x)(d^k-1)} dx \leq \max\{1, |\Omega|\} \max\{\|\hat{u}_R\|_{p(x)d^k}^{p(x_k)d^k}, \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k}\},$$

$$(4.6) \quad \int_{\Omega} \hat{u}_R |\hat{u}_R|^{\alpha(x)-1} |\hat{v}_R|^{\beta(x)+1} |\hat{u}_R|^{1+p(x)(d^k-1)} dx \leq 2 \max\{\|\hat{u}_R\|_{p(x)d^k}^{p(x_k)d^k}, \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k}\},$$

where $|\Omega|$ denotes the Lebesgue measure of a set Ω in \mathbb{R}^N .

Proof. Before starting the proof, let us note that

$$\begin{aligned} \frac{\alpha(x)+1+p(x)(d^k-1)}{p(x)d^k} + \frac{\beta(x)+1}{q(x)d^k} &= \left[\frac{\alpha(x)+1}{p(x)} + \frac{\beta(x)+1}{q(x)} \right] \frac{1}{d^k} + \frac{d^k-1}{d^k} \\ &= \frac{1}{d^k} + \frac{d^k-1}{d^k} = 1, \end{aligned}$$

where d is chosen as in (4.1).

Let us prove (4.5). Since $\hat{u}_R \in L^{p(x)d^k}(\Omega)$ and $p(x)d^k > p(x)d^k - p(x) + 1 > 0$ then $\hat{u}_R \in L^{\frac{p(x)d^k}{1+p(x)(d^k-1)}}(\Omega)$. Therefore, by Hölder's inequality and Mean value Theorem, there exist x_k and $t_k \in \Omega$ such that

$$\begin{aligned} \int_{\Omega} \hat{u}_R^{1+p(x)(d^k-1)} dx &\leq \|1_{\Omega}\|_{d^k p'(x)} \|\hat{u}_R\|_{d^k p(x)}^{d^k p(x_k)} = \|1_{\Omega}\|_{d^k p'(x)}^{\frac{d^k p'(t_k)}{d^k p'(x)}} \|\hat{u}_R\|_{d^k p(x)}^{d^k p(x_k)} \\ &= |\Omega|^{\frac{1}{d^k p'(t_k)}} \|\hat{u}_R\|_{d^k p(x)}^{d^k p(x_k)} \leq \max\{1, |\Omega|\} \|\hat{u}_R\|_{d^k p(x)}^{d^k p(x_k)}. \end{aligned}$$

This shows that the inequality (4.5) holds true. Here p' and p are conjugate variable exponents functions.

Next, we show (4.6). By (4) and Young's inequality, we get

$$\begin{aligned} (4.7) \quad & \left| \int_{\Omega} |\hat{u}_R|^{\alpha(x)+1+p(x)(d^k-1)} |\hat{v}_R|^{\beta(x)+1} dx \right| \leq \int_{\Omega} |\hat{u}_R|^{\alpha(x)+1+p(x)(d^k-1)} |\hat{v}_R|^{\beta(x)+1} dx \\ & \leq \int_{\Omega} \frac{\alpha(x)+1+p(x)(d^k-1)}{p(x)d^k} |\hat{u}_R|^{p(x)d^k} dx + \int_{\Omega} \frac{\beta(x)+1}{q(x)d^k} |\hat{v}_R|^{q(x)d^k} dx \\ & \leq \int_{\Omega} |\hat{u}_R|^{p(x)d^k} dx + \int_{\Omega} |\hat{v}_R|^{q(x)d^k} dx. \end{aligned}$$

Observe from (3.13) that

$$\int_{\Omega} \left| \frac{\hat{u}_R}{\|\hat{u}_R\|_{p(x)d^k}} \right|^{p(x)d^k} dx = 1.$$

Using the mean value theorem, there exists $x_k \in \Omega$ such that

$$(4.8) \quad \int_{\Omega} |\hat{u}_R|^{p(x)d^k} dx = \|\hat{u}_R\|_{p(x)d^k}^{p(x_k)d^k}.$$

Similarly, we can find $y_k \in \Omega$ such that

$$(4.9) \quad \int_{\Omega} |\hat{v}_R|^{q(x)d^k} dx = \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k}.$$

Then, combining (4.7), (4.8) and (4.9), the inequality (4.6) holds true, ending the proof of the lemma 6. ■

By using the Lemma we can prove the next result.

Lemma 7. *Assume (H.1)-(H.4) hold. Let $(\hat{u}_R, \hat{v}_R) \in X_0^{p(x), q(x)}(\Omega)$ be a solution of problem (1.1). Then,*

$$(\hat{u}_R, \hat{v}_R) \in L^{p(x)d^k}(\Omega) \times L^{q(x)d^k}(\Omega), \forall k \in \mathbb{N}.$$

Proof. We employ a recursive reasoning. Since $(\hat{u}_R, \hat{v}_R) \in X_0^{p(x), q(x)}(\Omega)$, it is obvious that $(\hat{u}_R, \hat{v}_R) \in L^{p(x)}(\Omega) \times L^{q(x)}(\Omega)$. So, (4.5) remains true for $k = 0$.

Assume that the conjecture “ $(\hat{u}_R, \hat{v}_R) \in L^{p(x)d^l}(\Omega) \times L^{q(x)d^l}(\Omega)$ ” holds at every level $l \leq k$ and we claim that

$$(4.10) \quad (\hat{u}_R, \hat{v}_R) \in L^{p(x)d^{k+1}}(\Omega) \times L^{q(x)d^{k+1}}(\Omega).$$

To do it, we inserte $\varphi = \hat{u}_R^{1+p(x)(d^k-1)}$ in (4.4) we get

$$(4.11) \quad \begin{aligned} & \int_{\Omega} |\nabla \hat{u}_R|^{p(x)-2} \nabla \hat{u}_R \nabla (\hat{u}_R^{1+p(x)(d^k-1)}) dx \\ &= \lambda_R^* \int_{\Omega} c(x)(\alpha(x) + 1) \hat{u}_R |\hat{u}_R|^{\alpha(x)-1} |\hat{v}_R|^{\beta(x)+1} \hat{u}_R^{1+p(x)(d^k-1)} dx. \end{aligned}$$

Observe that

$$(4.12) \quad \begin{aligned} & \int_{\Omega} |\nabla \hat{u}_R|^{p(x)-2} \nabla \hat{u}_R \nabla (\hat{u}_R^{1+p(x)(d^k-1)}) dx \\ &= \int_{\Omega} (d^k - 1) \nabla p \nabla \hat{u}_R |\nabla \hat{u}_R|^{p(x)-2} \hat{u}_R^{1+p(x)(d^k-1)} \ln \hat{u}_R dx \\ & \quad + \int_{\Omega} [1 + p(x)(d^k - 1)] |\nabla \hat{u}_R|^{p(x)} \hat{u}_R^{p(x)(d^k-1)} dx. \end{aligned}$$

and

$$(4.13) \quad |\nabla \hat{u}_R|^{p(x)} \hat{u}_R^{p(x)(d^k-1)} = \frac{1}{d^{kp(x)}} |\nabla (\hat{u}_R)^{d^k}|^{p(x)} \hat{u}_R$$

Then on the one hand

$$(4.14) \quad \begin{aligned} & \int_{\Omega} \frac{1+p(x)(d^k-1)}{d^{kp(x)}} |\nabla (\hat{u}_R)^{d^k}|^{p(x)} dx \geq \int_{\Omega} \frac{d^k}{d^{kp(x)}} |\nabla (\hat{u}_R)^{d^k}|^{p(x)} dx \\ & \geq \frac{1}{d^{k(p^+-1)}} \int_{\Omega} |\nabla (\hat{u}_R)^{d^k}|^{p(x)} dx, \end{aligned}$$

on the other hand, since \hat{u}_R is assumed of class $C^1(\overline{\Omega})$ and taking $\sup_{x \in \Omega} |\nabla p| = M_p < +\infty$, we have

$$(4.15) \quad \int_{\Omega} (d^k - 1) |\nabla p| |\nabla \hat{u}_R|^{p(x)-1} \hat{u}_R^{1+p(x)(d^k-1)} |\ln \hat{u}_R| dx \leq \hat{C} M_p \int_{\Omega} \hat{u}_R^{1+p(x)(d^k-1)} dx,$$

with some constant $\hat{C} > 0$. Hence, gathering (4.11), (4.12), (4.14) and (4.15) together, one has

$$(4.16) \quad \begin{aligned} & \int_{\Omega} |\nabla (\hat{u}_R)^{d^k}|^{p(x)} dx \leq d^{k(p^+-1)} \int_{\Omega} [1 + p(x)(d^k - 1)] |\nabla \hat{u}_R|^{p(x)} \hat{u}_R^{p(x)(d^k-1)} dx \\ & \leq d^{k(p^+-1)} \int_{\Omega} (d^k - 1) |\nabla \hat{u}_R|^{p(x)-1} |\nabla p| \hat{u}_R^{1+p(x)(d^k-1)} |\ln \hat{u}_R| dx \\ & \quad + \lambda_R^* \|c\|_{\infty} (\alpha^+ + 1) d^{k(p^+-1)} \int_{\Omega} |\hat{u}_R|^{\alpha(x)+1+p(x)(d^k-1)} |\hat{v}_R|^{\beta(x)+1} dx \\ & \leq \hat{C}_p d^{k(p^+-1)} \left[\int_{\Omega} \hat{u}_R^{1+p(x)(d^k-1)} dx + \right. \\ & \quad \left. \lambda_R^* \|c\|_{\infty} (\alpha^+ + 1) \int_{\Omega} |\hat{u}_R|^{\alpha(x)+1+p(x)(d^k-1)} |\hat{v}_R|^{\beta(x)+1} dx \right], \end{aligned}$$

where $\hat{C}_p = \max\{1, \hat{C} M_p\}$.

Thanks to the use of the hypothesis (H.3), the embeddings $L^{\pi_p(x)}(\Omega) \hookrightarrow L^{dp(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\pi_p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{dp(x)}(\Omega)$ are continuous and thus, for any $z \in W_0^{1,p(x)}(\Omega)$. We can conclude that there exists a constant $K > 0$ so that

$$(4.17) \quad \|z\|_{p(x)d} \leq K \|z\|_{1,p(x)}.$$

From (3.13) and through the mean value theorem observe that there exists $\xi_k \in \Omega$ such that

$$\begin{aligned} 1 &= \int_{\Omega} \left| \frac{|\hat{u}_R|}{\|\hat{u}_R\|_{p(x)d^{k+1}}} \right|^{p(x)d^{k+1}} dx \\ &= \int_{\Omega} \left| \frac{|\hat{u}_R|^{d^k}}{\|\hat{u}_R\|_{p(x)d}^{d^k}} \right|^{p(x)d} \times \left(\frac{\|\hat{u}_R\|_{p(x)d}^{d^k}}{\|\hat{u}_R\|_{p(x)d^{k+1}}^{d^k}} \right)^{p(x)d} dx = \left(\frac{\|\hat{u}_R\|_{p(x)d}^{d^k}}{\|\hat{u}_R\|_{p(x)d^{k+1}}^{d^k}} \right)^{p(\xi_k)d}, \end{aligned}$$

which leads to

$$(4.18) \quad \|\hat{u}_R\|_{p(x)d}^{d^k} = \|\hat{u}_R\|_{p(x)d^{k+1}}^{d^k}.$$

Recalling from (2.9) that for every $z \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$

$$(4.19) \quad \int_{\Omega} \left| \frac{|\nabla z|}{\|z\|_{1,p(x)}} \right|^{p(x)} dx = 1.$$

Applying (4.17) and (4.19) to $z = \hat{u}_R^{d^k}$, besides the mean value theorem and (4.18), there exists $x_k \in \Omega$ such that

$$(4.20) \quad \begin{aligned} K^{p(x_k)} \int_{\Omega} |\nabla(\hat{u}_R)^{d^k}|^{p(x)} dx &= K^{p(x_k)} \|\hat{u}_R\|_{1,p(x)}^{p(x_k)} = K^{p(x_k)} \|\hat{u}_R\|_{1,p(x)}^{p(x_k)} \\ &\geq \|\hat{u}_R\|_{dp(x)}^{p(x_k)} = \|\hat{u}_R\|_{p(x)d^{k+1}}^{p(x_k)} = (\|\hat{u}_R\|_{p(x)d^{k+1}}^{p(x_k)d^{k+1}})^{\frac{1}{d}}. \end{aligned}$$

Combining (4.16), (4.20) with Lemma 6, we get the following estimate

$$(4.21) \quad \|\hat{u}_R\|_{p(x)d^{k+1}}^{p(x_k)d^{k+1}} \leq C_1 d^{kd(p^+-1)} \left(\max\{\|\hat{u}_R\|_{p(x)d^k}^{p(x_k)d^k}, \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k}\} \right)^d,$$

Acting also in (4.4) with $\psi = \hat{v}^{1+q(x)(d^k-1)}$ and repeating the argument above, we obtain

$$(4.22) \quad \|\hat{v}_R\|_{q(x)d^{k+1}}^{q(x_k)d^{k+1}} \leq C_2 d^{kd(q^+-1)} \left(\max\{\|\hat{v}_R\|_{p(x)d^k}^{p(x_k)d^k}, \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k}\} \right)^d,$$

where C_1 and C_2 are two strictly positive constants.

So, it derives

$$(4.23) \quad \begin{aligned} \max\{\|\hat{u}_R\|_{p(x)d^{k+1}}^{p(x_{k+1})d^{k+1}}, \|\hat{v}_R\|_{q(x)d^{k+1}}^{q(y_{k+1})d^{k+1}}\} &\leq C_3 d^{k\hat{d}} \left(\max\{\|\hat{v}_R\|_{p(x)d^k}^{p(x_k)d^k}, \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k}\} \right)^d \\ &\leq C_3 d^{k\hat{d}} \left(\max\{\|\hat{v}_R\|_{p(x)d^k}^{p(x_k)d^k}, \|\hat{v}_R\|_{q(x)d^k}^{q(y_k)d^k}\} \right)^d, \end{aligned}$$

where \hat{d} satisfies (4.2) and $C_3 = \max\{C_1, C_2\}$.

Before continuing, we distinguish the cases where $\|\hat{u}_R\|_{p(x)d^{k+1}}$, $\|\hat{v}_R\|_{q(x)d^{k+1}}$, $\|\hat{u}_R\|_{p(x)d^k}$ and $\|\hat{v}_R\|_{q(x)d^k}$ are each either less than one or either greater than one. Using (H.4) and (4.1) we obtain

$$(4.24) \quad \begin{aligned} \ln \left(\max\{\|\hat{u}_R\|_{p(x)d^{k+1}}^{d^{k+1}}, \|\hat{v}_R\|_{q(x)d^{k+1}}^{d^{k+1}}\} \right) &\leq \ln(C_3 d^{k\hat{d}}) \\ &\quad + \hat{d} \ln \left(\max\{\|\hat{u}_R\|_{p(x)d^k}^{d^k}, \|\hat{v}_R\|_{q(x)d^k}^{d^k}\} \right). \end{aligned}$$

Now set

$$(4.25) \quad E_k = \max \left\{ \ln \|\hat{u}_R\|_{p(x)d^k}^{d^k}, \ln \|\hat{v}_R\|_{q(x)d^k}^{d^k} \right\} \quad \text{and} \quad \rho_k = ak + b,$$

with

$$(4.26) \quad a = \ln d^{\hat{d}}, \quad b = \ln C_3.$$

Then the recursive rule (4.24) becomes

$$(4.27) \quad E_{k+1} \leq \rho_k + \hat{d}E_k,$$

which in turn gives

$$(4.28) \quad E_{k+1} \leq E\hat{d}^k,$$

where

$$(4.29) \quad E = E_1 + \frac{b}{\hat{d}-1} + \frac{a\hat{d}}{(\hat{d}-1)^2}.$$

Indeed, using (4.27), (4.25) and Lemma 3, we get

$$(4.30) \quad \begin{aligned} E_{k+1} &\leq \rho_k + \hat{d}E_k \leq E_{k+1} \leq \rho_k + \hat{d}\rho_{k-1} + \hat{d}^2E_{k-1} \\ &\leq \rho_k + \hat{d}\rho_{k-1} + \hat{d}^2\rho_{k-2} + \hat{d}^3E_{k-2} \\ &\dots \\ &\leq \sum_{i=0}^{k-1} \hat{d}^i \rho_{k-i} + \hat{d}^k E_1 = \hat{d}^k (a \sum_{i=1}^k \frac{i}{\hat{d}^i} + b \sum_{i=1}^k \frac{1}{\hat{d}^i} + E_1) \\ &\leq \hat{d}^k (\frac{a\hat{d}}{(\hat{d}-1)^2} + \frac{b}{\hat{d}-1} + E_1) = \hat{d}^k E. \end{aligned}$$

Here Lemma 3 is applied choosing $s = 1/\hat{d} < 1$ and $r = k+1$. So on, according to (4.25) and (4.28), it follows that

$$(4.31) \quad \max\{\|\hat{u}_R\|_{p(x)d^k}, \|\hat{v}_R\|_{q(x)d^k}\} \leq e^{E \frac{\max\{p^+, q^+\}^{k-1}}{\hat{d}}}.$$

We fix k in \mathbb{N} , then we conclude that the assert (7) in Lemma 7 holds. The proof of Lemma 7 is complete. ■

Now, let us end the proof of Theorem by showing that (\hat{u}_R, \hat{v}_R) is bounded in Ω .

Lemma 8. *Let (\hat{u}_R, \hat{v}_R) be a solution of (1.1) corresponding to the eigenvalue λ^* . Assume that hypotheses (H.1)-(H.4) hold. Then, \hat{u}_R and \hat{v}_R are bounded in Ω .*

Proof. Argue by contradiction. It means that we suppose that for all $L > 0$, there exists $\Omega_L \subset \Omega$, $|\Omega_L| > 0$ such that for all $x \in \Omega_L$ we have $|\hat{u}_R(x)| > L$. Fix k and choose L large enough so that

$$(4.32) \quad \frac{p^- \ln L}{p^+ E \max\{p^+, q^+\}^{k+1}} > 1.$$

From lemma 1 we get

$$\begin{aligned} L^{p^- d^{k+1}} |\Omega_L| &\leq \int_{\Omega_L} L^{p(x)d^{k+1}} dx \leq \int_{\Omega_L} |\hat{u}_R|^{p(x)d^{k+1}} dx \\ &\leq \int_{\Omega} |\hat{u}_R|^{p(x)d^{k+1}} dx \leq \max\{\|\hat{u}_R\|_{p(x)d^{k+1}}^{p^+ d^{k+1}}, \|\hat{u}_R\|_{p(x)d^{k+1}}^{p^- d^{k+1}}\}. \end{aligned}$$

By (4.25), (4.28), and (4.1) it follows that

$$d^{k+1} p^- \ln L + \ln |\Omega_L| \leq p^+ E_{k+1} \leq p^+ E \hat{d}^k$$

After using (4.32) and dividing by \hat{d}^{k+1} , we get

$$(4.33) \quad 1 + \frac{\ln |\Omega_L|}{\hat{d}^{k+1}} < 1/\hat{d}.$$

We choose k sufficiently large in (4.33). This forces $\hat{d} < 1$, which contradicts (4.2). This proves the lemma 8. ■

Next, we show that \hat{u}_R and \hat{v}_R are strictly positive in Ω .

Lemma 9. *Let (\hat{u}_R, \hat{v}_R) be a solution of (1.1) corresponding to the eigenvalue λ^* . Then, the following asserts hold*

- (1) $\hat{u}_R > 0$ (resp. $\hat{v}_R > 0$) in Ω .
- (2) There exists $\delta \in (0, 1)$ such that \hat{u}_R is of class $C^{1,\delta}(\overline{\Omega})$.

Proof. Step 1. $\hat{u}_R \geq 0$ (resp. $\hat{v}_R \geq 0$ in Ω)

First, observe that

$$|u| = \max(u, 0) + \min(u, 0) \in W_0^{1,p(x)}(\Omega)$$

and

$$|\nabla|u|| \leq |\nabla \max(u, 0)| + |\nabla \min(u, 0)| \leq |\nabla u|.$$

Then it turns out that

$$\mathcal{A}(|\hat{u}_R|, |\hat{v}_R|) \leq \mathcal{A}(\hat{u}_R, \hat{v}_R) \text{ and } \mathcal{B}(|\hat{u}_R|, |\hat{v}_R|) = \mathcal{B}(\hat{u}_R, \hat{v}_R) = R.$$

Thereby (2.6) and (3.15), it follows that

$$\mathcal{A}(|\hat{u}_R|, |\hat{v}_R|) \leq \mathcal{A}(\hat{u}_R, \hat{v}_R) = R\lambda_R^* \leq \mathcal{A}(|\hat{u}_R|, |\hat{v}_R|),$$

which implies that $\mathcal{A}(|\hat{u}_R|, |\hat{v}_R|) = R\lambda_R^*$, showing that $(|\hat{u}_R|, |\hat{v}_R|)$ is a solution of (1.1). Therefore, we can assume that $\hat{u}_R, \hat{v}_R \geq 0$ in Ω .

Step 2. $\hat{u}_R > 0$ (resp. $\hat{v}_R > 0$) in Ω

Inspired by the ideas in [15], let $m > 0$ be a constant such that $h(\cdot) \in C^2(\overline{\partial\Omega}_{3m})$, with $\overline{\partial\Omega}_{3m} = \{x \in \overline{\Omega} : h(x) \leq 3m\}$. Define the functions

$$\mathcal{U}(x) = \begin{cases} e^{\kappa h(x)} - 1 & \text{if } h(x) < \sigma_1 \\ e^{\kappa h(x)} - 1 + \kappa e^{\kappa \sigma_1} \int_{\sigma_1}^{h(x)} \left(\frac{2m-t}{2m-\sigma_1}\right)^{\frac{2}{p^-}-1} dt & \text{if } \sigma_1 \leq h(x) < 2\sigma_1 \\ e^{\kappa h(x)} - 1 + \kappa e^{\kappa \sigma_1} \int_{\sigma_1}^{2m} \left(\frac{2m-t}{2m-\sigma_1}\right)^{\frac{2}{p^-}-1} dt & \text{if } 2\sigma_1 \leq h(x) \end{cases}$$

and

$$\mathcal{V}(x) = \begin{cases} e^{\kappa h(x)} - 1 & \text{if } h(x) < \sigma_2 \\ e^{\kappa h(x)} - 1 + \kappa e^{\kappa \sigma_2} \int_{\sigma_2}^{h(x)} \left(\frac{2m-t}{2m-\sigma_2}\right)^{\frac{2}{q^-}-1} dt & \text{if } \sigma_2 \leq h(x) < 2\sigma_2 \\ e^{\kappa h(x)} - 1 + \kappa e^{\kappa \sigma_2} \int_{\sigma_2}^{2m} \left(\frac{2m-t}{2m-\sigma_2}\right)^{\frac{2}{q^-}-1} dt & \text{if } 2\sigma_2 \leq h(x), \end{cases}$$

where $(\sigma_1, \sigma_2) = (\frac{\ln 2}{\kappa p^+}, \frac{\ln 2}{\kappa q^+})$ and $\kappa > 0$ is a parameter. A quite similar calculations as in [15, pages 11 and 12] furnish

$$(4.34) \quad -\Delta_{p(x)}(\mu_1 \mathcal{U}) \leq \lambda_R^* c(x)(\alpha(x) + 1)(\mu_1 \mathcal{U})^{\alpha(x)} \hat{v}_R^{\beta(x)+1} \text{ in } \Omega$$

and

$$(4.35) \quad -\Delta_{q(x)}(\mu_2 \mathcal{V}) \leq \lambda_R^* c(x)(\beta(x) + 1) \hat{u}_R^{\alpha(x)+1} (\mu_2 \mathcal{V})^{\beta(x)} \text{ in } \Omega,$$

where $\mu_1 = \exp(\kappa \frac{1-p^-}{\max_{\overline{\Omega}} |\nabla p| + 1})$ and $\mu_2 = \exp(\kappa \frac{1-q^-}{\max_{\overline{\Omega}} |\nabla q| + 1})$, provided that $\kappa > 0$ is large enough.

Now, for any $(z, w) \in X_0^{p(x), q(x)}(\Omega)$, denote by

$$\mathcal{L}_p(z, w) = -\Delta_{p(x)} z - \lambda_R^* c(x)(\alpha(x) + 1) |z|^{\alpha(x)-1} |w|^{\beta(x)+1}$$

and

$$\mathcal{L}_q(z, w) = -\Delta_{q(x)} w - \lambda_R^* c(x)(\beta(x) + 1) |z|^{\alpha(x)+1} |w|^{\beta(x)-1},$$

(4.34) and (4.35) may be formulated respectively as follows

$$\mathcal{L}_p(\mu_1 \mathcal{U}, \hat{v}_R) \leq 0 \text{ and } \mathcal{L}_q(\hat{u}_R, \mu_2 \mathcal{V}) \leq 0, \text{ in } \Omega.$$

Hence, from the above notation, we get

$$\mathcal{L}_p(\mu_1\mathcal{U}, \hat{v}_R) \leq 0 \leq \mathcal{L}_p(\hat{u}_R, \hat{v}_R) \quad \text{in } \Omega$$

and

$$\mathcal{L}_q(\hat{u}_R, \mu_2\mathcal{V}) \leq 0 \leq \mathcal{L}_q(\hat{u}_R, \hat{v}_R) \quad \text{in } \Omega.$$

Since $\mu_1\mathcal{U} = \hat{u}_R = 0$ and $\mu_2\mathcal{V} = \hat{v}_R = 0$ on $\partial\Omega$, we are allowed to apply [19, Lemma 2.3] and we deduce that

$$\hat{u}_R \geq \mu_1\mathcal{U} > 0 \quad \text{and} \quad \hat{v}_R \geq \mu_2\mathcal{V} > 0 \quad \text{in } \Omega.$$

Thereby the positivity of (\hat{u}_R, \hat{v}_R) in Ω is proven.

To end the proof of Lemma 8, we claim a regularity property for \hat{u}_R and \hat{v}_R .

Step 3. Regularity property

For $p, q \in C^1(\overline{\Omega}) \cap C^{0,\theta}(\overline{\Omega})$ for certain $\theta \in (0, 1)$, owing to [6, Theorem 1.2] the solution (\hat{u}_R, \hat{v}_R) belongs to $C^{1,\delta}(\overline{\Omega}) \times C^{1,\delta}(\overline{\Omega})$ for certain $\delta \in (0, 1)$. This completes the proof. ■

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